Examen de fin d'études secondaires
Section $\boxed{B}$. Math 2. le 10 juin 2004.

$\boxed{A}$ I

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = (x+1)^2 \cdot e^{-x}$$

1°. $\text{dom} f = \mathbb{R}$; $\forall x \in \mathbb{R} : f(x) \geq 0 \Rightarrow \mathcal{G}_f$ au-dessus de (Ox) ; pas A.V.

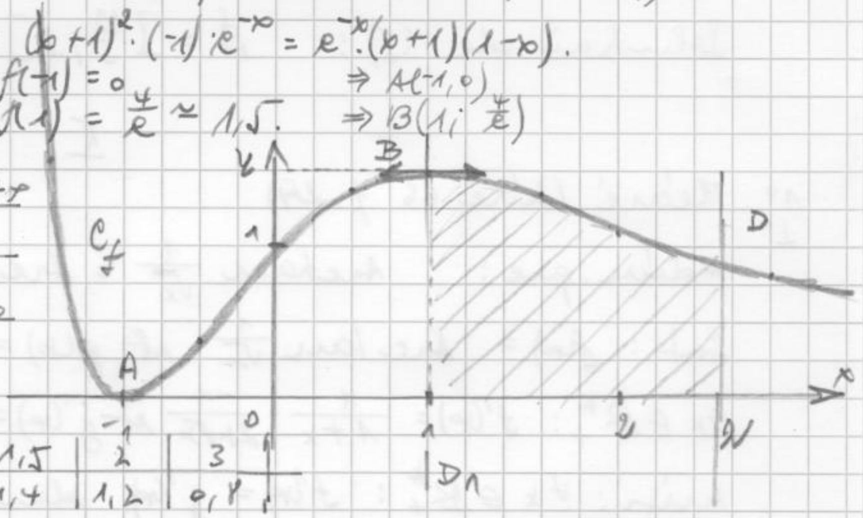
- $\lim\limits_{x \to +\infty} f(x) = 0$. A.H. $y = 0$.

$\lim\limits_{x \to -\infty} f(x) = +\infty$; $\lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to -\infty} \frac{x+2+\frac{1}{x}}{e^x} = +\infty$ pas d'A.O.

$\mathcal{G}_f$ admet une branche parabolique de dir. asympt. (Oy).

- $\forall x \in \mathbb{R} : f'(x) = 2(x+1)^1 \cdot 1 e^{-x} + (x+1)^2 \cdot (-1) \cdot e^{-x} = e^{-x}(x+1)(1-x)$.
- $f'(x) = 0 \Leftrightarrow \begin{cases} x = -1 \Rightarrow f(-1) = 0 & \Rightarrow A(-1, 0) \\ x = 1 \Rightarrow f(1) = \frac{4}{e} \simeq 1{,}5 & \Rightarrow B(1 ; \frac{4}{e}) \end{cases}$
- Tableau de variation.

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$		$\searrow$	0



- Tableau de valeurs :

x	$-1{,}5$	$-0{,}5$	0	$0{,}5$	$1{,}5$	2	3
$f(x)$	$-1{,}1$	$0{,}4$	1	$1{,}4$	$1{,}4$	$1{,}2$	$0{,}8$

2° a) f étant continue sur $\mathbb{R}$ admet des primitives sur $\mathbb{R}$.

b) $F(x) = \int f(x)\,dx = \int (x+1)^2 e^{-x}\,dx$ $\qquad u(x) = (x+1)^2 \quad v'(x) = e^{-x}$

$\qquad u'(x) = 2(x+1) \quad v(x) = -e^{-x}$

$F(x) = -(x+1)^2 \cdot e^{-x} + 2\int (x+1) \cdot e^{-x}\,dx$ $\qquad u(x) = x+1$ idem

$\qquad u'(x) = 1$

$F(x) = -(x+1)^2 \cdot e^{-x} + 2\left[-(x+1) \cdot e^{-x} + \int e^{-x}\,dx\right]$

$F(x) = -(x+1)^2 \cdot e^{-x} - 2(x+1) \cdot e^{-x} - 2 \cdot e^{-x} = -(x+1)^2 \cdot e^{-x} - 2(x+2) \cdot e^{-x} + k$.

condition : $F(1) = 0 \Leftrightarrow -\frac{4}{e} - \frac{6}{e} + k = 0 \Rightarrow k = \frac{10}{e}$

la primitive particulière cherchée est : $F(x) = -(x+1)^2 e^{-x} - 2(x+2) \cdot e^{-x} + \frac{10}{e}$.

c) $\mathcal{A}(\lambda) = \int_1^\lambda f(x)\,dx = F(\lambda)$ unités d'aires (car $f(x) \geq 0, \forall x \in \mathbb{R}$).

$\mathcal{A}(\lambda) = (-\lambda^2 - 4\lambda - 5) \cdot e^{-\lambda} + \frac{10}{e}$.

d) $\lim\limits_{\lambda \to +\infty} \mathcal{A}(\lambda) = \lim\limits_{\lambda \to +\infty} \left[(-\lambda^2 - 4\lambda - 5) \cdot e^{-\lambda} + \frac{10}{e}\right] = \frac{10}{e}$ unités d'aire.

$\boxed{B}$ 1° $\qquad$ (E) $\qquad e^{3x} = -k^2 \cdot e^x + 2k \cdot e^{2x}$ $\qquad (k \in \mathbb{R})$

!

(E) $\quad e^x(e^{2x} - 2k \cdot e^x + k^2) = 0 \quad | : e^x \neq 0$

(E) $\quad (e^{2x} - 2k \cdot e^x + k^2) = (e^x - k)^2 = 0$

(E) $\quad e^x = k$

Discussion : $\bullet$ $k \in]-\infty, 0]$: (E) est impossible et $S = \Phi$.

$\bullet$ $k \in]0 ; +\infty[$: $S = \{\ln k\}$.

! (E) existe $\forall x \in \mathbb{R}$.

2° (J) $\ln 24 + \ln(3-x) < \ln(x+1) + \ln(25x-49)$

Existence de (J) : $3-x>0 \iff x<3$
$x+1>0 \iff x>-1$
$25x-49>0 \iff x>\frac{49}{25}$

Finalement : (J) existe $\forall x \in \left]\frac{49}{25}; 3\right[$.

Résolution de (J) : $\ln[24(3-x)] < \ln(x+1)(25x-49)$

$\iff 24(3-x) < (x+1)(25x-49)$ car $x \mapsto \ln x$ est une bijection $\nearrow$ sur $\mathbb{R}_+^*$

$\iff 25x^2 - 121 > 0$

$\iff x \in \left]-\infty; -\frac{11}{5}\right[\cup \left]\frac{11}{5}; +\infty\right[$.

Solution de (J) : $S = \left]\frac{11}{5}; 3\right[$

II

1° Théorie (livre 66 p. 29)

2° Adém. que : $\operatorname{Arctan}\frac{1}{\sqrt{x}} = \operatorname{Arccot}\sqrt{x}$ $(\forall x \in \mathbb{R}_+^*)$.

Soit : $f(x) = \operatorname{Arctan}\frac{1}{\sqrt{x}}$ et $g(x) = \operatorname{Arccot}\sqrt{x}$

$\forall x \in \mathbb{R}_+^*$: $f'(x) = \frac{1}{1+x}\cdot\frac{-1}{2x\sqrt{x}}$ et $g'(x) = -\frac{+1}{1+x}\cdot\frac{1}{2\sqrt{x}}$

Ainsi : $\forall x \in \mathbb{R}_+^*$: $f'(x) = g'(x)$ donc $f(x) = g(x) + C$

Pour $x=1$: $f(1) = g(1) + C$ remplaçons $\operatorname{Arctan} 1 = \operatorname{Arccot} 1 + C$
$\frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow C = 0$.

Par conséquent : $\operatorname{Arctan}\frac{1}{\sqrt{x}} = \operatorname{Arccot}\sqrt{x}$ c.q.f.d.

3° $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto y = \operatorname{Arctan}\frac{1-x^2}{5x-x^2}$

- $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \operatorname{Arctan} 1 = \frac{\pi}{4}$.
- $\lim_{x\to 1} f(x) = \lim_{x\to1} \operatorname{Arctan}\frac{1-x^2}{5x-x^2} = \operatorname{Arctan} 0 = 0 = f(1)$.
- $\lim_{x\to0} f(x) = \lim_{x\to0} \operatorname{Arctan}\frac{1-x^2}{x(5-x)} = \operatorname{Arctan}\frac{1}{5\cdot 0}$

$\lim_{x\to0^+} f(x) = \frac{1}{5\cdot 0^+} = +\frac{\pi}{2}$; $\lim_{x\to0^-} f(x) = \frac{1}{5\cdot 0^-} = -\frac{\pi}{2}$

- $\lim_{x\to1}\frac{f(x)}{x-1} = \lim_{x\to1}\frac{\operatorname{Arctan}\frac{1-x^2}{5x-x^2}}{x-1} = \frac{0}{0}$ forme indéterminée.

Règle de l'Hôpital :

$$f'(x) = \frac{1}{1+\left(\frac{1-x^2}{5x-x^2}\right)^2}\cdot\frac{-2x(5x-x^2)-(1-x^2)(5-2x)}{(5x-x^2)^2}$$

$$f'(x) = \frac{1}{1+\left(\frac{1-x^2}{5x-x^2}\right)^2}\cdot\frac{-10x^2+\cancel{2x^3}-5+2x+5x^2-\cancel{2x^3}}{(5x-x^2)^2} = \frac{1}{1+\left(\frac{1-x^2}{5x-x^2}\right)^2}\cdot\frac{-5x^2+2x-5}{(5x-x^2)^2}$$

donc : $$\lim_{x\to1}\frac{f(x)}{x-1} = \lim_{x\to1}\frac{1}{1+\left(\frac{1-x^2}{5x-x^2}\right)^2}\cdot\frac{-5x^2+2x-5}{(5x-x^2)^2} = \frac{1}{1+0}\cdot\frac{-8}{16} = -\frac{1}{2}$$

1° $J = \int \frac{2x^3+x^2+2x+2}{x^4+3.x^2+2}\,dx$

Posons: $f(x) = \frac{2x^3+x^2+2x+2}{x^4+3x^2+2} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+2} = \frac{(ax+b)(x^2+2)+(cx+d)(x^2+1)}{(x^2+1)(x^2+2)}$

$\Leftrightarrow\ 2x^3+x^2+2x+2 = ax^3+2ax+bx^2+2b+cx^3+cx+dx^2+d$

$$\Leftrightarrow \begin{cases} a + c = 2 & \Rightarrow c = 2-a \\ b + d = 1 & \Rightarrow b = 1-d \\ 2a + c = 2 & \Rightarrow 2a+2-a = 2 \Rightarrow a = 0 \\ 2b + d = 2 & \Rightarrow 2-2d+d = 2 \Rightarrow d = 0 \end{cases} \qquad \begin{matrix} \Rightarrow c = 2 \\ \Rightarrow b = 1. \end{matrix}$$

donc: $f(x) = \frac{1}{x^2+1} + \frac{2x}{x^2+2} \Rightarrow J = \int \frac{1}{1+x^2}\,dx + \int \frac{2x}{x^2+2}\,dx$

$J = \operatorname{Arctan} x + \ln(x^2+2) + k.$

2° a) $J = \int \frac{x^2}{1+x^2}\,dx = \int \frac{(x^2+1)-1}{1+x^2}\,dx = \int \left[1 - \frac{1}{1+x^2}\right]dx$

$J = x - \operatorname{Arctan} x + k.$

b.) $K = \int \frac{x^2}{1+x^2}\cdot \operatorname{Arctan} x \cdot dx$ $\quad u(x) = \operatorname{Arctan} x$, $u'(x) = \frac{1}{1+x^2}$ $\quad v'(x) = \frac{x^2}{1+x^2}$, $v(x) = x - \operatorname{Arctan} x$

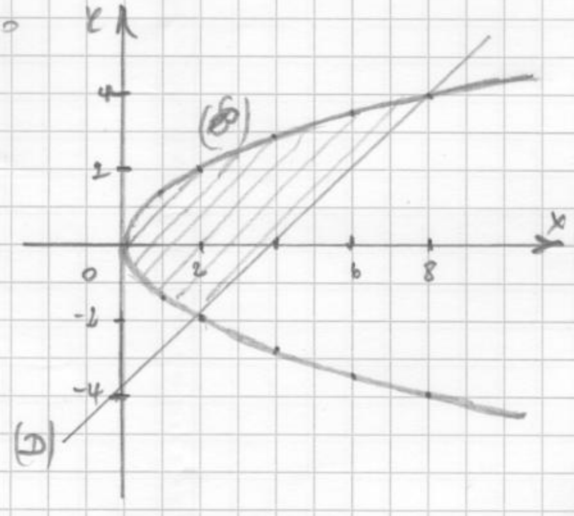
$K = (x - \operatorname{Arctan} x)\operatorname{Arctan} x - \int \frac{x - \operatorname{Arctan} x}{1+x^2}\,dx$

$K = x\cdot\operatorname{Arctan} x - \operatorname{Arctan}^2 x - \int \frac{x\,dx}{1+x^2} + \int \frac{\operatorname{Arctan} x}{1+x^2}\,dx$

$K = x\cdot\operatorname{Arctan} x - \operatorname{Arctan}^2 x - \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\operatorname{Arctan}^2 x + k$

$K = x\cdot\operatorname{Arctan} x - \frac{1}{2}\operatorname{Arctan}^2 x - \frac{1}{2}\ln(1+x^2) + k.$

3°



$$\begin{cases} y^2 = 2x & P \\ y = x - 4 & D \end{cases}$$

$P \cap D$: $(x-4)^2 = 2x$
$x^2 - 10x + 16 = 0$
$x = 2$ ou $x = 8$

Donc: $P \cap D = \{A(2,-2);\ B(8,4)\}$.

Pour calculer cette aire, il faut calculer 2 intégrales; par des raisons de symétrie de la surface correspondant aux $x \in [0,2]$, on a:

$\mathcal{A} = 2\int_0^2 (+\sqrt{2x})\,dx + \int_2^8 [\sqrt{2x} - (x-4)]\,dx$

$\mathcal{A} = 2\sqrt{2}\cdot\frac{2}{3}[x\sqrt{x}]_0^2 + \sqrt{2}\cdot\frac{2}{3}[x\sqrt{x}]_2^8 - \frac{1}{2}[x^2]_2^8 + 4[x]_2^8$

$\mathcal{A} = \frac{4}{3}\sqrt{2}\cdot 2\sqrt{2} + \frac{2}{3}\sqrt{2}(16\sqrt{2} - 2\sqrt{2}) - \frac{1}{2}(64-4) + 4\cdot 6$

$\mathcal{A} = \frac{16}{3} + \frac{64}{3} - \frac{8}{3} - 30 + 24 = \frac{72}{3} - 6 = 24 - 6 = 18$ unités d'aire.

4° $y = f(x) = \frac{1}{3}\sqrt{x}.(3-x)$

$\operatorname{dom} f = \mathbb{R}_+$

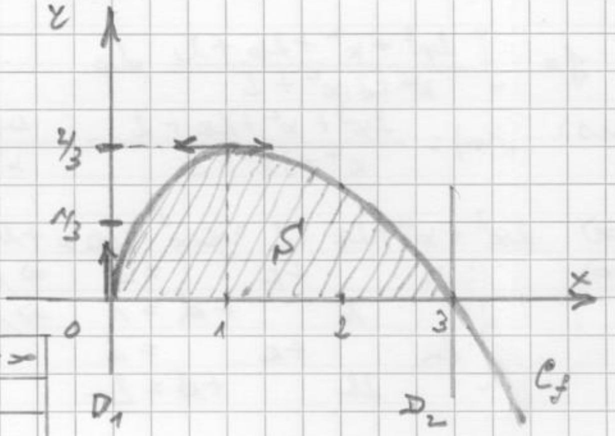
$f(x) = 0 \Leftrightarrow x = 0$ ou $x = 3$.

$f'(x) = \frac{1}{3}\cdot\left[\frac{1}{2\sqrt{x}}(3-x) + \sqrt{x}(-1)\right]$

$f'(x) = \frac{1}{3}\cdot\frac{3-x-2x}{2\sqrt{x}} = \frac{1-x}{2\sqrt{x}}$

$f'(x) = 0 \Leftrightarrow x = 1$ et $f(1) = \frac{2}{3}$.

x	0		1		$+\infty$
$f'(x)$	$\|$	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{2}{3}$	$\searrow$	



volume : $V = \pi\int_0^3 f^2(x)\,dx = \frac{\pi}{9}\int_0^3 x(3-x)^2\,dx$

$V = \frac{\pi}{9}\int_0^3 (x^3 - 6x^2 + 9x)\,dx = \frac{\pi}{9}\left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^3$

$V = \frac{\pi}{9}\left(\frac{81}{4} - 54 + \frac{81}{2}\right) = \frac{\pi}{9}\cdot 27\left(\frac{3}{4} - 2 + \frac{3}{2}\right) = 3\pi\left(\frac{9}{4} - 2\right)$

$V = \frac{3\pi}{4}$ unités de volume.