Examen de fin d'études secondaires.
Section [B]. Math. 2. Juin 2003. (1)

I

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 2x - |x| \cdot \ln x^2 & \text{si } x \neq 0 \\ 0 & \text{si } x = 0. \end{cases}$$

1) $\forall x > 0:\ f(x) = 2x - x\cdot\ln x^2 = 2x - 2x\ln x = 2x(1-\ln x)$
$\forall x < 0:\ f(x) = 2x + x\cdot\ln x^2 = 2x + 2x\ln(-x) = 2x\cdot[1+\ln(-x)]$.

* $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}[2x - 2x\cdot\ln x] = \lim\limits_{x\to 0^+} 2\cdot\dfrac{1-\ln x}{\frac{1}{x}} = 2\cdot\lim\limits_{x\to 0^+}\dfrac{-\frac{1}{x}}{-\frac{1}{x^2}}$
$= 2\cdot\lim\limits_{x\to 0^+} x = 2\cdot 0 = 0 = f(0) \Rightarrow f$ est cont. à droite en $x_0 = 0$.

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}[2x + 2x\ln(-x)] = 2\cdot\lim\limits_{x\to 0^-}\dfrac{1+\ln(-x)}{\frac{1}{x}} = 2\cdot\lim\limits_{x\to 0^-}\dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$
$= 2\cdot\lim\limits_{x\to 0^-}(-x) = 0 = f(0) \Rightarrow f$ est continue à gauche en $x_0 = 0$.

? $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 0 = f(0) \Rightarrow f$ est continue en $x_0 = 0$.

* $\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+}\dfrac{2\cancel{x}(1-\ln x)}{\cancel{x}} = -(-\infty) = +\infty$
$\lim\limits_{x\to 0^-}\dfrac{f(x)}{x} = \lim\limits_{x\to 0^-}\dfrac{2\cancel{x}[1+\ln(-x)]}{\cancel{x}} = -\infty$
f n'est pas dér. en $x_0 = 0$.

2) $\text{dom}\, f = \text{dom}_c f = \text{dom}\, f' = \mathbb{R}^*$.

3) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} 2x(1-\ln x) = (+\infty)(-\infty) = -\infty$.
$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x} = \lim\limits_{x\to+\infty} 2\cdot(1-\ln x) = -\infty$: B.P. de D.A. (Oy).

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} 2x[1+\ln(-x)] = (-\infty)(+\infty) = -\infty$.
$\lim\limits_{x\to-\infty}\dfrac{f(x)}{x} = 2\cdot\lim\limits_{x\to-\infty}[1+\ln(-x)] = +\infty$: B.P. de D.A (Oy).

4) $G \cap (Ox)$: $2x - |x|\cdot\ln x^2 = 0$
- $x > 0$: $2x(1-\ln x) = 0 \Leftrightarrow x = e$
- $x < 0$: $2x[1+\ln(-x)] = 0 \Leftrightarrow \ln(-x) = -1 \Leftrightarrow x = -\frac{1}{e}$.
- et $f(0) = 0$

d'où 3 points d'intersection avec (Ox) : $O(0,0)$; $A(e,0)$; $B(-\frac{1}{e}, 0)$

5) $\forall x > 0$: $f'(x) = 2 - 2(\ln x + x\cdot\frac{1}{x}) = -2\cdot\ln x$; $f'(x) = 0 \Leftrightarrow x = 1$ et $f(1) = 2$.
$\forall x < 0$: $f'(x) = 2 + 2[\ln(-x) + x\cdot\frac{1}{x}] = 2\cdot\ln(-x) + 4$.
$f'(x) = 0 \Leftrightarrow \ln(-x) = -2 \Leftrightarrow x = -\frac{1}{e^2}$ et $f(-\frac{1}{e^2}) = \frac{2}{e^2} = 0{,}27$.

Tableau des variations

x	$-\infty$		$-\frac{1}{e^2} \simeq -0{,}14$		0		1		$+\infty$
$f'(x)$		$+$	0	$-$	$\Vert$	$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	$2/e^2$	$\searrow$	0	$\nearrow$	2	$\searrow$	$-\infty$

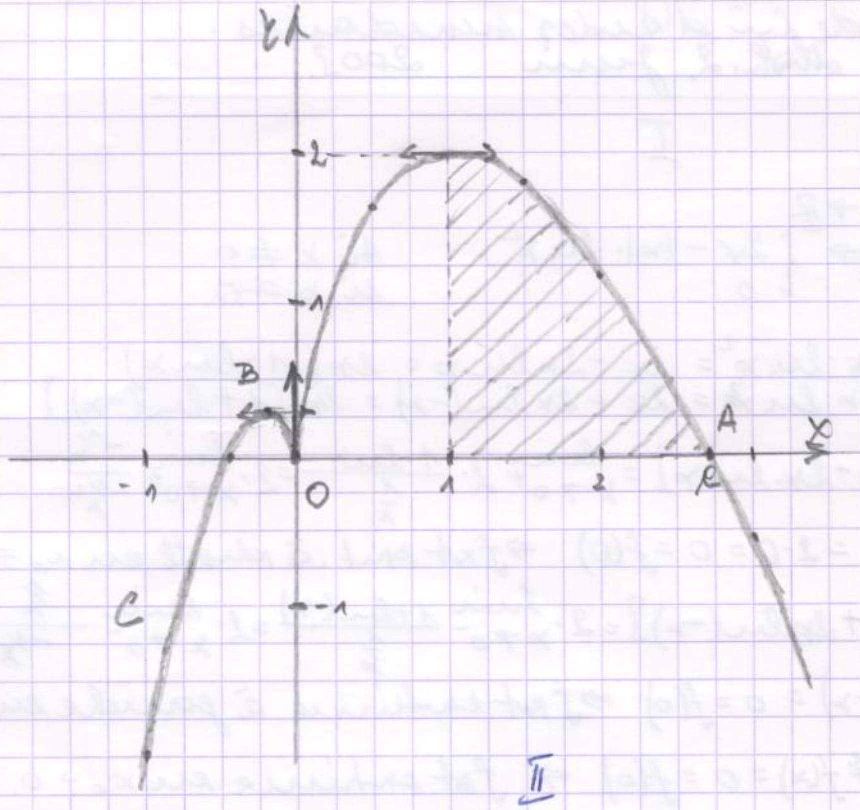
car : $\forall x < 0$: $f'(x) \geq 0 \Leftrightarrow \ln(-x) \geq -2 \Leftrightarrow -x \geq \frac{1}{e^2} \Leftrightarrow x \leq -\frac{1}{e^2}$.

6) $\forall x > 0$: $f''(x) = -\frac{2}{x} < 0$
$\forall x < 0$: $f''(x) = \frac{2}{x} < 0$

x		0	
$f''(x)$	$-$	$\Vert$	$-$
$f(x)$	$\cap$	0	$\cap$

G n'admet pas de points d'inflexion.

7)



$f(1/2) = 1{,}69$
$f(1{,}5) = 1{,}78$
$f(2) = 1{,}23$
$f(3) = -0{,}59$
$f(-1) = -2$

8)

$$\mathcal{A} = \int_1^e (2x - 2x\ln x)\,dx$$

$$\mathcal{A} = \left[x^2\right]_1^e - 2\left[\tfrac{1}{2}x^2 \ln x\right]_1^e - \tfrac{1}{2}\int_1^e x\,dx$$

$u(x) = \ln x \qquad v'(x) = x$
$u'(x) = \frac{1}{x} \qquad v(x) = \frac{1}{2}x^2$

$$\mathcal{A} = e^2 - 1 - \tfrac{1}{2}e^2 - \tfrac{1}{2}$$

$$\mathcal{A} = \left(\tfrac{1}{2}e^2 - \tfrac{3}{2}\right) \times (2 \times 2)\ cm^2$$

$$\mathcal{A} = 2e^2 - 6\ cm^2.$$

II

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin\left(\tfrac{1}{2}\operatorname{Arc}\cos x\right)$$

1) Existence : $-1 \le x \le 1 \Rightarrow \operatorname{dom} f = [-1, 1]$.
Or : $0 \le \operatorname{Arc}\cos x \le \pi \Rightarrow 0 \le \frac{1}{2}\operatorname{Arc}\cos x \le \frac{\pi}{2}$.
$\Rightarrow f(x) \in [0, 1] \Rightarrow \operatorname{Im} f : [0, 1]$.

Posons : $\frac{1}{2}\cdot\operatorname{Arc}\cos x = y \Leftrightarrow \operatorname{Arc}\cos x = 2y \Leftrightarrow x = \cos 2y$ avec $y \in [0, \frac{\pi}{2}]$
Donc : $x = 1 - 2\sin^2 y$
D'où :
$$\sin^2 y = \frac{1-x}{2} \geq 0$$
$$\sin y = \sqrt{\frac{1-x}{2}} \quad \text{car } y \in [0, \tfrac{\pi}{2}]$$

P.c. $f : [-1, 1] \to [0, 1]$
$$x \mapsto \sqrt{\frac{1-x}{2}}$$

2) Calculons : $\operatorname{Arc}\cot \frac{1}{3} + \operatorname{Arc}\cot \frac{1}{2}$.

Posons : $\operatorname{Arc}\cot \frac{1}{3} = a \Leftrightarrow \cot a = \frac{1}{3}$ avec $a, b \in]0, \frac{\pi}{2}[$
$\operatorname{Arc}\cot \frac{1}{2} = b \Leftrightarrow \cot b = \frac{1}{2}$ car $\cot a > 0$ et $\cot b > 0$

Formule : $$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b} = \frac{\frac{1}{\cot a} + \frac{1}{\cot b}}{1 - \frac{1}{\cot a}\cdot\frac{1}{\cot b}} = \frac{3+2}{1-3\cdot 2} = -1$$

Comme : $\cot(a+b) = \dfrac{1}{\tan(a+b)} = -1 \Leftrightarrow a+b = \operatorname{Arc}\cot(-1) = \frac{3\pi}{4}$.

Finalement : $\operatorname{Arc}\cot \frac{1}{3} + \operatorname{Arc}\cot \frac{1}{2} = \frac{3\pi}{4}$.

3) $f(x) = \operatorname{Arc}\tan \dfrac{x+a}{1-ax} \quad (a \in \mathbb{R}^*)$

a) $\operatorname{dom} f = \operatorname{dom} f' = \mathbb{R} - \left\{\frac{1}{a}\right\}$.

$$f'(x) = \frac{1}{1 + \left(\frac{x+a}{1-ax}\right)^2} \cdot \frac{1(1-ax) - (x+a)(-a)}{(1-ax)^2} = \frac{\cancel{(1-ax)^2}(1 \cancel{-ax} \cancel{+ax} + a^2)}{[(1-ax)^2 + (x+a)^2]\cdot\cancel{(1-ax)^2}}$$

$$f'(x) = \frac{1+a^2}{1-\cancel{2ax}+a^2x^2+x^2+\cancel{2ax}+a^2} = \frac{1+a^2}{(1+x^2)+a^2(1+x^2)} = \frac{1+a^2}{(1+x^2)(1+a^2)}$$

$f'(x) = \dfrac{1}{1+x^2}$ (expression indépendante de a).

b.) $\quad \operatorname{Arctan}\dfrac{x-1}{1+x} + \operatorname{Arctan} x = \dfrac{\pi}{4}. \qquad (E)$

Existence : (E) existe $\forall x \in \mathbb{R} - \{-1\}$.
Résolution :
Posons : $\operatorname{Arctan}\dfrac{x-1}{1+x} = a \iff \dfrac{x-1}{1+x} = \tan a$ et $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$

$\operatorname{Arctan} x = b \iff x = \tan b$ et $-\dfrac{\pi}{2} < b < \dfrac{\pi}{2}$

(E) s'écrit : $a + b = \dfrac{\pi}{4}$.

$$\Rightarrow \tan(a+b) = \tan\frac{\pi}{4} = 1$$

$$\iff \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b} = 1$$

$$\iff \frac{\frac{x-1}{1+x} + x}{1 - \frac{x-1}{1+x} \cdot x} = 1$$

$$\iff \frac{x-1+x+x^2}{1+x-x^2+x} = 1$$

$$\iff \frac{x^2+2x-1}{-x^2+2x+1} = 1$$

avec : $x^2-2x-1 \neq 0$ c.à.d. $x \notin \{1-\sqrt{2} ; 1+\sqrt{2}\}$.

$$\iff x^2 + \cancel{2x} - 1 = -x^2 + \cancel{2x} + 1$$

$$\iff 2x^2 - 2 = 2(x^2-1) = 2(x+1)(x-1) = 0$$

$$\iff x = 1 \quad \text{car } x = -1 \text{ est à écarter.}$$

Si $x = 1+\sqrt{2}$ alors (E) $\rightarrow \operatorname{Arctan}\dfrac{\sqrt{2}}{2+\sqrt{2}} + \operatorname{Arctan}(1+\sqrt{2}) \simeq 1,57 \neq 0,79$.

Si $x = 1-\sqrt{2}$ alors (E) $\rightarrow \operatorname{Arctan}\dfrac{-\sqrt{2}}{2-\sqrt{2}} + \operatorname{Arctan}(1-\sqrt{2}) \simeq -1,57 \neq 0,79$

Finalement : $\quad S = \{1\}$.

II

1) $\quad A = \displaystyle\int_{\pi/4}^{\pi/3} \frac{dx}{\sin x \cdot \cos^2 x} \qquad u(x) = \frac{1}{\sin x} \qquad v'(x) = \frac{1}{\cos^2 x}$

$\qquad u'(x) = -\dfrac{\cos x}{\sin^2 x} \qquad v(x) = \tan x$

$$A = \left[\frac{\tan x}{\sin x}\right]_{\pi/4}^{\pi/3} + \int_{\pi/4}^{\pi/3} \frac{\tan x \cdot \cos x}{\sin^2 x}\, dx = \left[\frac{1}{\cos x}\right]_{\pi/4}^{\pi/3} + \int_{\pi/4}^{\pi/3} \frac{dx}{\sin x}$$

Or : $\displaystyle\int_{\pi/4}^{\pi/3} \frac{dx}{\sin x} = \int_{\pi/4}^{\pi/3} \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}\, dx = \int_{\pi/4}^{\pi/3} \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2}\, dx$

$$= \left[\ln\left|\tan\frac{x}{2}\right|\right]_{\pi/4}^{\pi/3} = \ln\tan\frac{\pi}{6} - \ln\tan\frac{\pi}{8} = \ln\frac{1}{\sqrt{3}} - \ln\tan\frac{\pi}{8}$$

P.c. $\quad A = \dfrac{1}{\cos\frac{\pi}{3}} - \dfrac{1}{\cos\frac{\pi}{4}} + \ln\dfrac{1}{\sqrt{3}} - \ln\tan\dfrac{\pi}{8} = 2 - \sqrt{2} + \ln\dfrac{1}{\sqrt{3}} - \ln\tan\dfrac{\pi}{8}$

④ $B = \int_0^{1/2} e^{\operatorname{Arc}\cos x}\,dx$

2)

Ch^t de variable : $\operatorname{Arccos} x = t \iff x = \cos t \implies dx = -\sin t\,dt$

$x = 0 \iff t = \frac{\pi}{2}$; $x = \frac{1}{2} \iff t = \frac{\pi}{3}$.

$B = -\int_{\pi/2}^{\pi/3} e^t \sin t\,dt = \int_{\pi/3}^{\pi/2} e^t \sin t\,dt$ $\quad u(t) = \sin t \quad v'(t) = e^t$; $u'(t) = \cos t \quad v(t) = e^t$

$B = [e^t \sin t]_{\pi/3}^{\pi/2} - \int_{\pi/3}^{\pi/2} e^t \cos t\,dt$ $\quad u(t) = \cos t \quad v'(t) = e^t$; $u'(t) = -\sin t \quad v(t) = e^t$

$B = [e^t \sin t]_{\pi/3}^{\pi/2} - \left\{ [e^t \cos t]_{\pi/3}^{\pi/2} + \int_{\pi/3}^{\pi/2} e^t \sin t\,dt \right\}$

$B = [e^t \sin t]_{\pi/3}^{\pi/2} - [e^t \cos t]_{\pi/3}^{\pi/2} - B$

Donc : $2B = e^{\pi/2} - e^{\pi/3} \cdot \frac{\sqrt{3}}{2} - \left(0 - e^{\pi/3} \cdot \frac{1}{2}\right)$

$B = \frac{1}{2}\left[e^{\pi/2} - e^{\pi/3} \cdot \frac{\sqrt{3}-1}{2}\right]$.

3)

$I_1 = \int_0^1 \frac{e^x}{e^x+1}\,dx = [\ln(e^x+1)]_0^1 = \ln(1+e) - \ln 2$.

$I_n + I_{n+1} = \int_0^1 \left[\frac{e^{nx}}{e^x+1} + \frac{e^{(n+1)x}}{e^x+1}\right] dx = \int_0^1 \frac{e^{nx}(1+e^x)}{e^x+1}\,dx = \int_0^1 e^{nx}\,dx$

$= \frac{1}{n}[e^{nx}]_0^1 = \frac{1}{n}[e^n - 1]$

P.R. $\quad I_{n+1} = \frac{1}{n}(e^n - 1) - I_n$.

Alors $\quad I_2 = \frac{1}{1}(e-1) - I_1 = e - 1 - \ln\frac{e+1}{2}$ $\qquad (n=1)$

$I_3 = \frac{1}{2}(e^2-1) - I_2 = \frac{e^2-1}{2} - e + 1 + \ln\frac{e+1}{2}$ $\qquad (n=2)$

IV

$f : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto y = x^x$ $\qquad$ $g : \mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto y = x^{-x}$

1) $\text{dom } f = \text{dom } f' = \mathbb{R}_+^*$
$\text{dom } g = \text{dom } g' = \mathbb{R}_+^*$.

2) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} e^{x \ln x} = e^{+\infty} = +\infty$.

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{x \ln x} = \exp\left[\lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}}\right] = \exp\left[\lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}\right]$

$= e^{\lim\limits_{x \to 0^+}(-x)} = e^0 = 1.$

$\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} \dfrac{1}{x^x} = \dfrac{1}{+\infty} = 0^+$ $\qquad$ A.H. $y = 0$.

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \dfrac{1}{x^x} = \dfrac{1}{1} = 1$.

3° • $f'(x) = (x^x)' = (e^{x \ln x})' = e^{x \ln x}.(\ln x + 1)$

$f'(x) = 0 \iff \ln x = -1 \iff x = \frac{1}{e} \simeq 0{,}37$ et $f(\frac{1}{e}) = (\frac{1}{e})^{1/e} \simeq 0{,}69$.

• $g'(x) = (x^{-x})' = (e^{-x \ln x})' = e^{-x \ln x}.(-\ln x - 1)$.

$g'(x) = 0 \iff \ln x = -1 \iff x = \frac{1}{e}$ et $g(\frac{1}{e}) = (\frac{1}{e})^{-1/e} \simeq 1{,}44$.

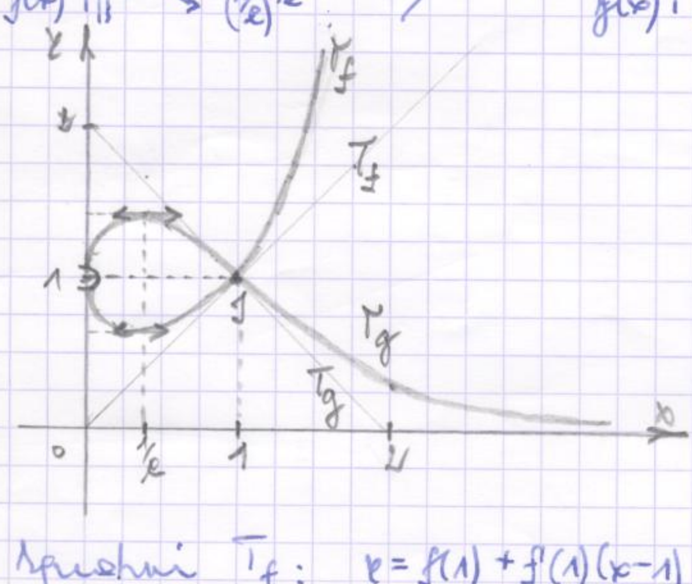
• Tableaux de variation.

x	0		$1/e$		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ 1	↘	$(\frac{1}{e})^{1/e}$	↗	$+\infty$

x	0		$1/e$		$+\infty$
$g'(x)$	‖	$+$	0	$-$	
$g(x)$	‖ 1	↗	$(\frac{1}{e})^{-1/e}$	↘	0

4°

5°



$f'(1) = e^0.(\ln 1 - 1) = 1$

$g'(1) = e^0.(-\ln 1 - 1) = -1$

$\Rightarrow f'(1) \cdot g'(1) = -1$.

(le produit des pentes des deux tangentes étant -1, elles sont perpendiculaires).

Equation T_f : $y = f(1) + f'(1)(x-1)$
$y = 1 + 1(x-1)$
$y = x$

Equation T_g : $y = g(1) + g'(1).(x-1)$
$y = 1 - 1(x-1)$
$y = 2 - x$.