I

1) voir livre p 86, 87

2) $\lim\limits_{+\infty} \dfrac{\log_3(2^x-3)}{x} = \lim\limits_{+\infty} \dfrac{\ln(2^x-3)}{\ln 3 \cdot x} \underset{(H)}{=} \lim\limits_{+\infty} \dfrac{\frac{\ln 2 \cdot 2^x}{2^x-3}\cdot 2^{-x}}{\ln 3}\cdot 2^{-x}$

$= \lim\limits_{+\infty} \dfrac{\ln 2}{\ln 3 \cdot (1-3 \cdot 2^{-x})} = \dfrac{\ln 2}{\ln 3} = \log_3 2$

3) $\lim\limits_{-\infty} \left(1-\frac{5}{x}\right)^{2+x}$ f.i. 1^{∞}

posons $h = -\frac{5}{x} \Leftrightarrow x = -\frac{5}{h} \Leftrightarrow 2+x = \frac{2h-5}{h}$

$x \to -\infty$ ssi $h \to 0$

D'où $\lim\limits_{x\to-\infty} \left(1-\frac{5}{x}\right)^{2+x} = \lim\limits_{h\to 0} \left[(1+h)^{\frac{1}{h}}\right]^{2h-5} = e^{-5}$

4) $2\ln(x+1) \leq \ln(x^3+1) - \ln x$ (*)

C.E. $\begin{cases} x+1>0 & (1) \\ x^3+1>0 & (2) \\ x>0 & (3) \end{cases}$ si (3) est vérifiée, alors (1) et (2) le sont aussi

$\Leftrightarrow x > 0$

$D_E = \mathbb{R}_+^*$

(*) $\Leftrightarrow \ln(x+1)^2 + \ln x \leq \ln(x^3+1)$

$\Leftrightarrow \cancel{\ln}\, x(x+1)^2 \leq \cancel{\ln}\,(x^3+1)$

$\Leftrightarrow x(x^2+2x+1) - x^3 - 1 \leq 0$

$\Leftrightarrow x^3+2x^2+x-x^3-1 \leq 0$

$\Leftrightarrow 2x^2+x-1 \leq 0$

x		-1		$\frac{1}{2}$	
$2x^2+x-1$	+	0	−	0	+

$\Delta = 1+8 = 9$, $x' = \frac{-1+3}{4} = \frac{1}{2}$, $x'' = \frac{-1-3}{4} = -1$

$\Leftrightarrow -1 \leq x \leq \frac{1}{2}$

$S = \left]0, \frac{1}{2}\right]$

II $f(x) = 5 - 5x^2e^x$

1) . $D_f = \mathbb{R} = D_{f'}$

• $\lim\limits_{-\infty}\left(5-5\,\underset{\substack{\downarrow\\+\infty}}{x^2}\,\underset{\substack{\downarrow\\0}}{e^x}\right)$ $+\infty\cdot 0$ f.i. $=\lim\limits_{-\infty}\left(5-5\dfrac{x^2\;{\scriptstyle\to+\infty}}{e^{-x}\;{\scriptstyle\to+\infty}}\right)\underset{(H)}{=}\lim\limits_{-\infty}\left(5-5\cdot\dfrac{2x\;{\scriptstyle\to-\infty}}{-e^{-x}\;{\scriptstyle\to-\infty}}\right)$

$\underset{(H)}{=}\lim\limits_{-\infty}\left(5-5\cdot\underbrace{\dfrac{2}{\underset{\to+\infty}{e^{-x}}}}_{\to 0}\right)=5$ $\quad$ A.H.G. $y=5$

$\lim\limits_{+\infty}\left(5-\underbrace{5\,\underset{\substack{\downarrow\\+\infty}}{x^2}\,\underset{\substack{\downarrow\\+\infty}}{e^x}}_{\to+\infty}\right)=-\infty$, pas d'A.H.D.

$\lim\limits_{+\infty}\dfrac{f(x)}{x}=\lim\limits_{+\infty}\left(\underset{\substack{\downarrow\\0}}{\dfrac{5}{x}}-5\,\underset{\substack{\downarrow\\+\infty}}{xe^x}\right)=-\infty$, B.p. de direction (Oy)

• $f'(x)=-10x\cdot e^x-5x^2\cdot e^x=-5x\,\underset{>0}{e^x}\,(2+x)$

x		-2		0	
$-5x$	$+$		$+$	0	$-$
$x+2$	$-$	0	$+$		$+$
$f'(x)$	$-$	0	$+$	0	$-$

x	$-\infty$		-2		0		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
f	5	$\searrow$	$5-20e^{-2}\simeq 2{,}3$	$\nearrow$	5	$\searrow$	$-\infty$

• $f''(x)=-10\cdot e^x-10x\cdot e^x-10x\cdot e^x-5x^2e^x=\underset{<0}{-5e^x}(x^2+4x+2)$

signe contraire de x^2+4x+2, $\Delta=16-8=8$, $x'=\dfrac{-4+2\sqrt2}{2}=\sqrt2-2\simeq-0{,}6$

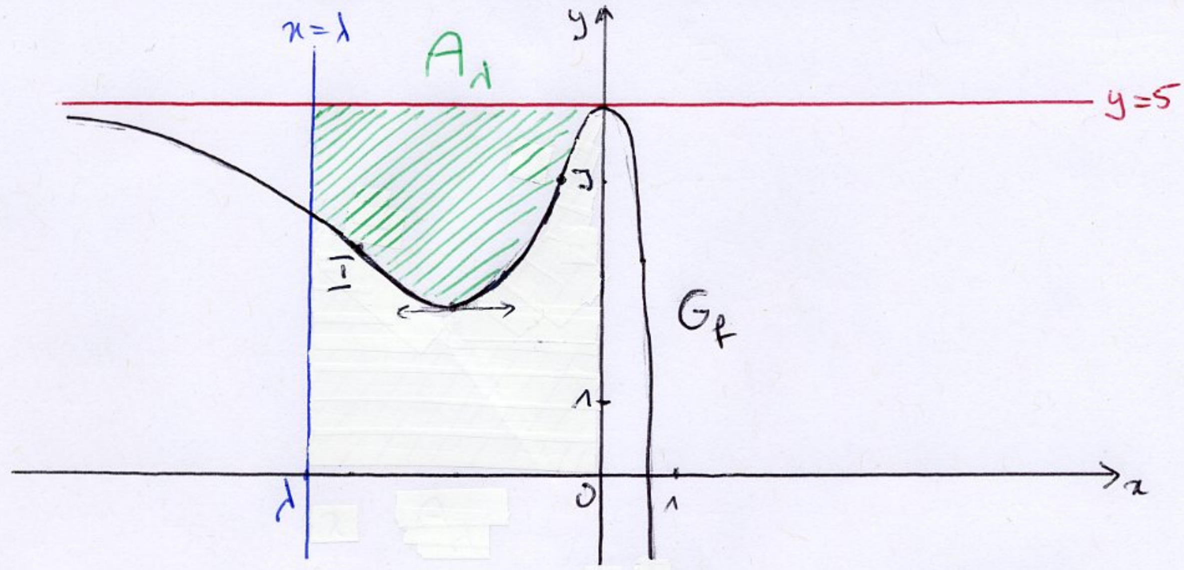
$x''=\dfrac{-4-2\sqrt2}{2}=-\sqrt2-2\simeq-3{,}4$

x		$-\sqrt2-2$		$\sqrt2-2$	
$f''(x)$	$-$	0	$+$	0	$-$
G_f	$\frown$		$\smile$		$\frown$

2 pts d'inflexion: $I(-3{,}4\,;\,3{,}1)$

$J(-0{,}6\,;\,4)$

x	1	$0{,}5$
$f(x)$	$-8{,}6$	$2{,}9$



2) $A_\lambda = \int_\lambda^0 (5 - f(x))\,dx = \int_\lambda^0 (\cancel{5}-\cancel{5}+5x^2e^x)\,dx = \int_\lambda^0 \underbrace{5x^2e^x}_{g(x)}\,dx$

i.p.p. $u = 5x^2 \quad u' = 10x$
$\quad\; v' = e^x \quad v = e^x$

$G(x) = 5x^2e^x - \underbrace{\int 10xe^x\,dx}$

i.p.p. $u = 10x \quad u' = 10$
$\quad\; v' = e^x \quad v = e^x$

$G(x) = 5x^2e^x - \left[10xe^x - \int 10e^x\,dx\right]$
$= 5x^2e^x - 10xe^x + 10e^x$
$= 5e^x(x^2 - 2x + 2)$

$A_\lambda = G(0) - G(\lambda)$
$\underline{A_\lambda = 10 - 5e^\lambda(\lambda^2 - 2\lambda + 2)}$

$\lim\limits_{\lambda\to-\infty} A_\lambda = \lim\limits_{-\infty}\left(10 - 5\underset{\downarrow\atop 0}{e^\lambda}\cdot\underset{\downarrow\atop +\infty}{\lambda^2}\right) = \lim\limits_{-\infty}\left(10 - \frac{5\lambda^2 \to +\infty}{e^{-\lambda} \to +\infty}\right)$

$\underset{(H)}{=} \lim\limits_{-\infty}\left(10 + \frac{10\lambda \to -\infty}{+e^{-\lambda} \to +\infty}\right) \underset{(H)}{=} \lim\limits_{-\infty}\left(10 + \frac{10}{-e^{-\lambda}}\right) = \underline{\underline{10}}$

$\left(\frac{10}{-\infty}\right) = 0$

$\underline{\text{III}}$ 1) a) $\int_0^1 \frac{1-3x}{\sqrt{4-x^2}}\,dx = \int_0^1 \underbrace{\frac{1}{\sqrt{4-x^2}}}_{f(x)}\,dx - 3\cdot\int_0^1 \underbrace{\frac{x}{\sqrt{4-x^2}}}_{g(x)}\,dx$

$f(x) = \frac{1}{\sqrt{4(1-\frac{x^2}{4})}} = \frac{1}{2\sqrt{1-(\frac{x}{2})^2}}$

$\begin{cases} u = \frac{x}{2} \\ u' = \frac{1}{2} \\ f = \frac{u'}{\sqrt{1-u^2}} \\ F = \text{Asin}\, u \end{cases}$ $\quad F(x) = \text{Asin}\,\frac{x}{2}$

$g(x) = \dfrac{x}{\sqrt{4-x^2}}$ $\quad \begin{cases} u = 4-x^2 \\ u' = -2x \Leftrightarrow -\frac{1}{2}u' = x \\ g = -\frac{1}{2}\frac{u'}{\sqrt{u}} = -\frac{1}{2}u' \cdot u^{-\frac{1}{2}} \\ G = -\frac{1}{2} \cdot \dfrac{u^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{u} \end{cases}$ $\quad G(x) = -\sqrt{4-x^2}$

D'où $\displaystyle\int_0^1 \frac{1-3x}{\sqrt{4-x^2}}\,dx = \text{Asin}\,\frac{1}{2} - \text{Asin}\,0 - 3\cdot\left(-\sqrt{3}+2\right) = \frac{\pi}{6} + 3\sqrt{3} - 6$

b) $I = \displaystyle\int_0^1 \underbrace{(x^2-5)\cos \pi x}_{f(x)}\,dx$

i.p.p. $\quad u = x^2-5 \quad u' = 2x$

$\qquad v' = \cos \pi x \quad v = \frac{1}{\pi}\sin \pi x$

$F(x) = \frac{1}{\pi}(x^2-5)\sin \pi x - \frac{2}{\pi}\underbrace{\int x \sin \pi x\,dx}$

i.p.p. $\quad u = x \quad u' = 1$

$\qquad v' = \sin \pi x \quad v = -\frac{1}{\pi}\cos \pi x$

$F(x) = \frac{1}{\pi}(x^2-5)\sin \pi x - \frac{2}{\pi}\left[-\frac{1}{\pi}x\cos \pi x + \frac{1}{\pi}\int \cos \pi x\,dx\right]$

$= \frac{1}{\pi}\left[(x^2-5)\sin \pi x + \frac{2x}{\pi}\cos \pi x - \frac{2}{\pi}\cdot\frac{1}{\pi}\sin \pi x\right]$

$F(1) = \frac{1}{\pi}\left(-\frac{2}{\pi}\right) = -\frac{2}{\pi^2}$

$F(0) = 0$

D'où $\underline{I = -\frac{2}{\pi^2}}$

2) a) $(x-1)(x+1)(x^2+1) = (x^2-1)(x^2+1) = x^4-1$

$\dfrac{5x^3-3x^2+5x+5}{(x-1)(x+1)(x^2+1)} = \dfrac{a}{x-1} + \dfrac{b}{x+1} + \dfrac{c}{x^2+1} \quad (*)$

$(*)\,|\cdot(x-1) \Leftrightarrow \dfrac{5x^3-3x^2+5x+5}{(x+1)(x^2+1)} = a + \dfrac{b(x-1)}{x+1} + \dfrac{c(x-1)}{x^2+1}$

pour $x=1$: $\frac{12}{4} = a + 0 \Leftrightarrow \underline{a = 3}$

$(*)\,|\cdot(x+1) \Leftrightarrow \dfrac{\cdots}{(x-1)(x^2+1)} = \dfrac{a(x+1)}{x-1} + b + \dfrac{c(x+1)}{x^2+1}$

pour $x=-1$: $\dfrac{-8}{-2\cdot 2} = 0+b+0 \Leftrightarrow \underline{b = 2}$

$(*)\,|\cdot(x^2+1) \Leftrightarrow \dfrac{\cdots}{(x+1)(x-1)} = \dfrac{a(x^2+1)}{x-1} + \dfrac{b(x^2+1)}{x+1} + c$

pour $x=i$: $\dfrac{\cancel{5i^3}+3+\cancel{5i}+5}{-1-1} = 0+0+c \Leftrightarrow c = \frac{8}{-2} = -4$

D'où $f(x) = \frac{3}{x-1} + \frac{2}{x+1} - \frac{4}{x^2+1}$, $D_f =]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[$

b) Sur $I =]-1, 1[$: $F(x) = 3 \cdot \ln|x-1| + 2\ln|x+1| - 4 \cdot \text{Arctan}\, x + C$

$F(0) = 2 \Leftrightarrow 0 + C = 2 \Leftrightarrow C = 2$

D'où $F(x) = 3\ln|x-1| + 2\ln|x+1| - 4\,\text{Arctan}\, x + 2$

Problème :

1) a) On définit la fonction f

$f(x) = ax^3 + bx^2 + cx + d$

Conditions à vérifier :

$$\begin{cases} f(-6) = 2 \\ f(-4) = 0 \\ f(-1) = 3 \\ f(1) = 2 \end{cases}$$

V200 : $a = -\frac{1}{10}$ $b = -\frac{7}{10}$ $c = -\frac{2}{5}$ $d = \frac{16}{5}$

d'où $f(x) = -\frac{x^3}{10} - \frac{7x^2}{10} - \frac{2x}{5} + \frac{16}{5}$

b) $f'(x) = -\frac{3x^2}{10} - \frac{7x}{5} - \frac{2}{5}$

$f''(x) = -\frac{3x}{5} - \frac{7}{5}$

$f''(x) = 0 \Leftrightarrow x = -\frac{7}{3}$

on vérifie le changement de signe de $f''(x)$

x		$-7/3$	
$f''(x)$	$+$	0	$-$

$f(-\frac{7}{3}) = \frac{43}{27}$

Le point d'inflexion est $F(-\frac{7}{3} ; \frac{43}{27})$

c) on résout $f(x) = 2$

V200 : $x = -6$ ou $x = -2$ ou $x = 1$

Le chemin traverse à nouveau la piste cyclable en $G(-2 ; 2)$

d) Etudions la fonction h définie par $h(x) = f(x) - 2$ sur $[-6 ; 6]$

$h'(x) = f'(x) = -\frac{3x^2}{10} - \frac{7x}{5} - \frac{2}{5}$

on résout $h'(x) = 0$

V200 : $x_1 = \frac{-7 - \sqrt{37}}{3} \simeq -4,361$ $x_2 = \frac{-7 + \sqrt{37}}{3} \simeq -0,306$

x	-6		x_1		x_2		6
$h'(x)$		$-$	0	$+$	0	$-$	
$h(x)$	0	$\searrow$	$\simeq -2,075$	$\nearrow$	$\simeq 1,260$	$\searrow$	0

La distance maximale correspond au maximum de $|h(x)|$ sur $[-6 ; 6]$

$d_{max} \simeq 2,075$ km

2) On définit la fonction g

$g(x) = ax^2 + bx + c$

Conditions à vérifier :

$$\begin{cases} g(1) = 2 \\ g'(1) = f'(1) \\ g(6) = 2 \end{cases}$$

V200 : $a = \frac{21}{50}$ $b = -\frac{147}{50}$ $c = \frac{113}{25}$

d'où $g(x) = \frac{21x^2}{50} - \frac{147x}{50} + \frac{113}{25}$

3) $L_1 = \int_{-6}^{1} \sqrt{1 + f'(x)^2}\,dx$ $L_2 = \int_{1}^{6} \sqrt{1 + g'(x)^2}\,dx$

V200 $\simeq 10,093$ $\simeq 7,586$

$L = L_1 + L_2 \simeq 17,679$ km à 1 m près

4) $A_1 = \int_{-6}^{-2} (2 - f(x))\,dx = \frac{16}{3}$ $A_2 = \int_{-2}^{1} (f(x) - 2)\,dx = \frac{99}{40}$

$A_3 = \int_{1}^{6} (2 - g(x))\,dx = \frac{35}{4}$

$A_{totale} = A_1 + A_2 + A_3 = \frac{1987}{120}$ km^2

Prix $= A_{totale} \times 10^6 \times 1$

$= \frac{49\,675\,000}{3}$

$\simeq 16\,558\,333$ € à 1 € près

La commune ne peut pas accepter ce devis.