Corrigé C,D, math II

I 1) voir EM66 p 55

2) a) $\lim\limits_{0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim\limits_{0}\frac{x-e^x+1}{x(e^x-1)}=\left(\frac{0}{0}\right)$ f.i.

$\underset{(H)}{=}\lim\limits_{0}\frac{1-e^x}{1\cdot(e^x-1)+x\cdot e^x}=\left(\frac{0}{0}\right)$ f.i

$\underset{(H)}{=}\lim\limits_{0}\frac{-e^x}{e^x+1\cdot e^x+x\cdot e^x}=\frac{-1}{2}$

b) $\lim\limits_{+\infty}\left(\frac{2x+3}{2x-1}\right)^{3x+1}$ $\approx 3x \to +\infty$ f.i. 1^∞

$\approx \frac{2x}{2x}\to 1$

posons : $1+y=\frac{2x+3}{2x-1} \Leftrightarrow y=\frac{2x+3}{2x-1}-1 \Leftrightarrow y=\frac{2x+3-2x+1}{2x-1} \Leftrightarrow y=\frac{4}{2x-1}$

alors : $x\to+\infty$ ssi $y\to 0$

et $y=\frac{4}{2x-1} \Leftrightarrow 2x-1=\frac{4}{y} \Leftrightarrow 2x=\frac{4}{y}+1 \mid :2 \Leftrightarrow x=\frac{2}{y}+\frac{1}{2}$

donc $3x+1=\frac{6}{y}+\frac{3}{2}+1=\frac{6}{y}+\frac{5}{2}=\frac{12+5y}{2y}$

$\lim\limits_{x\to+\infty}\left(\frac{2x+3}{2x-1}\right)^{3x+1}=\lim\limits_{y\to 0}(1+y)^{\frac{12+5y}{2y}}=\lim\limits_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]^{\frac{12+5y}{2}}=e^6$

$(1+y)^{\frac{1}{y}} \to e$, $\frac{12+5y}{2}\to\frac{12}{2}=6$

3) a) $e^x<e^{-x}-2 \quad \mid \cdot e^x (>0)$

$\Leftrightarrow (e^x)^2<1-2e^x \quad (*)$

posons $y=e^x$, alors $(*) \Leftrightarrow y^2<1-2y \Leftrightarrow y^2+2y-1<0$

y		$-1-\sqrt{2}$		$\sqrt{2}-1$	
y^2+2y-1	+	0	−	0	+

$\Delta=4+4=8$, $y'=\frac{-2+\sqrt{8}}{2}=-1+\sqrt{2}$

$y''=\frac{-2-\sqrt{8}}{2}=-1-\sqrt{2}$

D'où $(*) \Leftrightarrow -1-\sqrt{2}<y<-1+\sqrt{2} \Leftrightarrow \underbrace{-1-\sqrt{2}<e^x}_{<0 \text{ vrai!}}<\sqrt{2}-1$

$\Leftrightarrow e^x<\sqrt{2}-1$

$\Leftrightarrow x<\ln\sqrt{2}-1$

$S=]-\infty, \ln(\sqrt{2}-1)[$

b) $\log_2\sqrt{2x+1}\le\log_4(5x-2)-\log_2 x \quad (*)$

C.E. $\begin{cases}2x+1>0\\5x-2>0\\x>0\end{cases} \Leftrightarrow \begin{cases}x>-\frac{1}{2}\\x>\frac{2}{5}\\x>0\end{cases}$ $D_E=]\frac{2}{5},+\infty)$

$(*) \Leftrightarrow \dfrac{\ln\sqrt{2x+1}}{\ln 2} \leq \dfrac{\ln(5x-2)}{\ln 4} - \dfrac{\ln x}{\ln 2}$ $\quad (\ln 4 = \ln 2^2 = 2\ln 2)$

$\Leftrightarrow \dfrac{1}{2}\dfrac{\ln(2x+1)}{\ln 2} \leq \dfrac{\ln(5x-2)}{2\ln 2} - \dfrac{\ln x}{\ln 2} \quad \Big| \cdot 2\ln 2 > 0$

$\Leftrightarrow \ln(2x+1) \leq \ln(5x-2) - 2\ln x$

$\Leftrightarrow \ln(2x+1) + \ln x^2 \leq \ln(5x-2)$

$\Leftrightarrow \cancel{\ln}\, x^2(2x+1) \leq \cancel{\ln}(5x-2)$

$\Leftrightarrow \underbrace{2x^3+x^2-5x+2}_{P(x)} \leq 0 \quad (*)$

$P(1)=0$ donc $P(x)$ div. par $x-1$:

	2	1	-5	2
1		2	3	-2
	2	3	-2	0

D'où $(*) \Leftrightarrow (x-1)(2x^2+3x-2) \leq 0$

x		-2		$\frac{2}{5}$ $\frac{1}{2}$		1	D_E
$x-1$	$-$		$-$		$-$	0	$+$
$2x^2+3x-2$	$+$	0	$-$	0	$+$		$+$
$P(x)$	$-$	0	$+$	0	$-$	0	$+$

$\Delta = 9+16 = 25$

$x' = \dfrac{-3+5}{4} = \dfrac{1}{2}$

$x'' = \dfrac{-3-5}{4} = -2$

$S = \left[\frac{1}{2}, 1\right]$

II

$f(x) = \dfrac{2x+2}{e^{x/2}} = 2(x+1)\cdot e^{-\frac{x}{2}}$

1) a) $e^{x/2} \neq 0$ donc $D_f = \mathbb{R}$

$\lim\limits_{-\infty} \dfrac{2x+2 \to -\infty}{e^{x/2} \to 0^+} = -\infty$, pas d'A.H.G.

$\lim\limits_{-\infty} \dfrac{f(x)}{x} = \lim\limits_{-\infty} \dfrac{2x+2}{x \cdot e^{x/2}}$ $\quad \approx \dfrac{2x}{x} \to 2$, $\; e^{x/2} \to 0^+$ $\quad = \left(\dfrac{2}{0+}\right) = +\infty$, D.p. de direction (Oy) pour $x \to -\infty$

$\lim\limits_{+\infty} \dfrac{2x+2}{e^{x/2}} = \left(\dfrac{+\infty}{+\infty}\right)$ f.i. $\underset{(H)}{=} \lim\limits_{+\infty} \dfrac{2}{\frac{1}{2}e^{x/2}} = 0$, $\frac{1}{2}e^{x/2} \to +\infty$, A.H.D. $y=0$

b) $f'(x) = \dfrac{2\cdot e^{x/2} - \frac{1}{2}e^{\frac{x}{2}}(2x+2)}{e^x} = \dfrac{2e^{x/2} - xe^{x/2} - e^{x/2}}{e^x} = \dfrac{e^{x/2} - xe^{x/2}}{e^x}$

$= \dfrac{e^{x/2}(1-x)}{e^x} = \dfrac{1-x}{e^{x/2}}$ signe de $1-x$

x	$-\infty$		1		$+\infty$
$f'(x)$		$+$	0	$-$	
f	$-\infty$	$\nearrow$	$\frac{4}{\sqrt{e}}$ $\approx 2{,}4$	$\searrow$	0

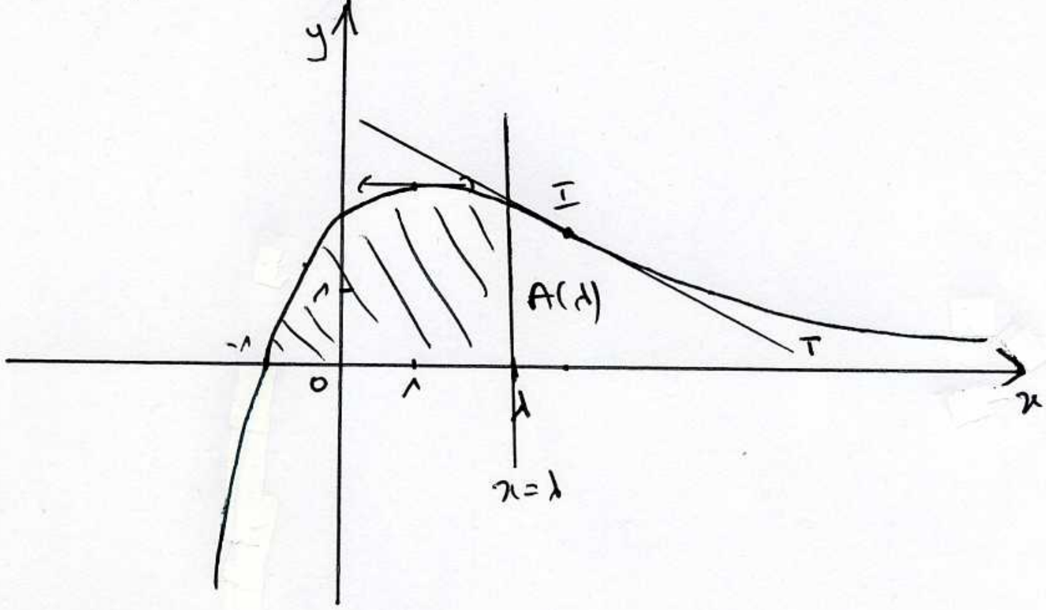
$$f''(x) = \frac{-1\,e^{x/2} - \frac{1}{2}e^{x/2}(1-x)}{e^x} = \frac{-\frac{3}{2}e^{\frac{x}{2}} + \frac{x}{2}e^{\frac{x}{2}}}{e^x} = \frac{\frac{1}{2}e^{\frac{x}{2}}(x-3)}{e^x}$$

$$= \frac{x-3}{2\,e^{x/2}}$$ signe de $x-3$

x	$-\infty$		3		$+\infty$
$f''(x)$		$-$	0	$+$	
G_f		$\cap$		$\cup$	

1 pt. d'inflexion : $\underline{I}(3\,;\approx 1{,}8)$

c)



d) $T : y - f(3) = f'(3)(x-3)$

$f'(3) = -\frac{2}{\sqrt{e}}$, $f(3) = \frac{8}{e^{3/2}}$

$T \equiv y = -\frac{2}{\sqrt{e}}(x-3) + \frac{8}{e^{3/2}} \equiv \underline{y = -\frac{2}{\sqrt{e}}x + \frac{6e+8}{e^{3/2}}}$

2) $A(\lambda) = \int_{-1}^{\lambda} f(x)\,dx = 2\int_{-1}^{\lambda} \underbrace{(x+1)e^{-\frac{x}{2}}}_{g(x)}\,dx$

i.p.p. $u = x+1$ $\quad u' = 1$
$\quad v' = e^{-\frac{x}{2}}$ $\quad v = -2e^{-\frac{x}{2}}$

$G(x) = -2e^{-\frac{x}{2}}(x+1) + 2\int e^{-\frac{x}{2}}dx = -2e^{-\frac{x}{2}}(x+1) - 4e^{-\frac{x}{2}} = -2e^{-\frac{x}{2}}(x+1+2)$

$= -2e^{-\frac{x}{2}}(x+3)$

$A(\lambda) = 2\cdot(G(\lambda) - G(-1))$

$= 2\left[-2e^{-\frac{\lambda}{2}}(\lambda+3) + 2\sqrt{e}\cdot 2\right]$

$= 4\left[2\sqrt{e} - e^{-\frac{\lambda}{2}}(\lambda+3)\right]$ cm^2

$\lim\limits_{\lambda\to+\infty} e^{-\frac{\lambda}{2}}(\lambda+3) = \lim\limits_{+\infty} \frac{\lambda+3}{e^{\frac{\lambda}{2}}} = \left(\frac{+\infty}{+\infty}\text{ f.i.}\right) \underset{(H)}{=} \lim\limits_{+\infty} \frac{1}{\frac{1}{2}e^{\frac{\lambda}{2}}} = \left(\frac{1}{+\infty}\right) = 0$

D'où : $\lim\limits_{\lambda\to+\infty} A(\lambda) = 8\sqrt{e}$

III 1) a) $\dfrac{1}{x(x-1)(x+1)} = \dfrac{a}{x} + \dfrac{b}{x-1} + \dfrac{c}{x+1} \quad (*)$

$(*)\,|\cdot x \iff \underset{\downarrow\,-1}{\dfrac{1}{(x-1)(x+1)}} = a + \underset{\downarrow\,0}{\dfrac{bx}{x-1}} + \underset{\downarrow\,0}{\dfrac{cx}{x+1}}$ pour $x \to 0$

donc $\underline{a=-1}$

$(*)\,|\,(x-1) \iff \underset{\downarrow\,\frac{1}{2}}{\dfrac{1}{x(x+1)}} = \underset{\downarrow\,0}{\dfrac{a(x-1)}{x}} + b + \underset{\downarrow\,0}{\dfrac{c(x-1)}{x+1}}$ pour $x \to 1$

donc $\underline{b=\frac{1}{2}}$

$(*)\,|\,(x+1) \iff \underset{\downarrow\,\frac{1}{2}}{\dfrac{1}{x(x-1)}} = \underset{\downarrow\,0}{\dfrac{a(x+1)}{x}} + \underset{\downarrow\,0}{\dfrac{b(x+1)}{x-1}} + c$ pour $x \to -1$

donc $c = \frac{1}{2}$

D'où $\dfrac{1}{x(x^2-1)} = -\dfrac{1}{x} + \dfrac{1}{2(x-1)} + \dfrac{1}{2(x+1)}$

b) posons: $u = x^2-1$

alors: $u' = 2x \iff \frac{1}{2}u' = x$

$f = \frac{1}{2}\,\dfrac{u'}{u^2} = \frac{1}{2}u^{-2}\cdot u'$ $\qquad$ donc $F(x) = -\dfrac{1}{2(x^2-1)} + k$

$F = \frac{1}{2}\,\dfrac{u^{-1}}{-1} = -\dfrac{1}{2u}$

c) $I = \displaystyle\int_2^3 \underbrace{\frac{x\ln x}{(x^2-1)^2}}_{g(x)}\,dx$ $\qquad$ i.p.p. $u = \ln x \qquad u' = \frac{1}{x}$

$v' = \dfrac{x}{(x^2-1)^2} \qquad v = -\dfrac{1}{2(x^2-1)}$ (d'après b)

$$G(x) = -\frac{\ln x}{2(x^2-1)} + \frac{1}{2}\int \frac{1}{x(x^2-1)}\,dx$$

$$= -\frac{\ln x}{2(x^2-1)} + \frac{1}{2}\int\left(-\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right)dx$$

$$= -\frac{\ln x}{2(x^2-1)} - \frac{1}{2}\ln|x| + \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1|$$

$$G(3) = -\frac{\ln 3}{16} - \frac{1}{2}\ln 3 + \frac{1}{4}\ln 2 + \frac{1}{4}\underbrace{\ln 4}_{=2\ln 2} = -\frac{9}{16}\ln 3 + \frac{3}{4}\ln 2$$

$$G(2) = -\frac{\ln 2}{6} - \frac{1}{2}\ln 2 + \cancel{\frac{1}{4}\ln 1} + \frac{1}{4}\ln 3 = -\frac{2}{3}\ln 2 + \frac{1}{4}\ln 3$$

$$I = G(3) - G(2) = -\frac{9}{16}\ln 3 + \frac{3}{4}\ln 2 + \frac{2}{3}\ln 2 - \frac{1}{4}\ln 3$$

$$\underline{I = \frac{17}{12}\ln 2 - \frac{13}{16}\ln 3}$$

2) $J = \int_0^{\frac{\pi}{2}} \underbrace{2\cos^2 x \sin x}\,dx + \int_0^{\frac{\pi}{2}} 4\sin^2 x\,dx = J_1 + J_2$

$u = \cos x$
$u' = -\sin x$
$f = -2u^2 u'$
$F = -2 \cdot \frac{u^3}{3}$
$F(x) = -\frac{2}{3}\cos^3 x$
$F(\frac{\pi}{2}) = 0$; $F(0) = -\frac{2}{3}$
$J_1 = 0 - (-\frac{2}{3}) = \frac{2}{3}$

$g(x) = \cancel{4}^{2}\,\frac{1-\cos 2x}{\cancel{2}} = 2 - 2\cos 2x$
$G(x) = 2x - \sin 2x$
$G(\frac{\pi}{2}) = \pi - 0 = \pi$
$G(0) = 0$
$J_2 = \pi - 0 = \pi$

D'où $J = \frac{2}{3} + \pi$

3) $A = \int_0^2 [2 - \underbrace{(x-1)^2} - \cos \pi x]\,dx = \left[2x - \frac{(x-1)^3}{3} - \frac{1}{\pi}\sin \pi x\right]_0^2$

$u = x-1$
$u' = 1$
$f = u^2 \cdot u'$
$F = \frac{u^3}{3}$

$= (4 - \frac{1}{3} - 0) - (0 - \frac{(-1)^3}{3} - 0)$
$= 4 - \frac{1}{3} - \frac{1}{3}$
$A = \frac{10}{3}$ u.a.

IV 1) $P(x) = x(x+80)(ax+b)$

$B(-60, 15) \in G_P$ donc $P(-60) = 15$ (1)
$C(-15, -10) \in G_P$ donc $P(-15) = -10$ (2)
donc (V200) : $a = \frac{71}{140\,400}$, $b = \frac{167}{9360}$

$P(x) = 0$, V200 : $x = 0$ ou $x = -80$ ou $x = c = -\frac{2505}{71}$

2) longueur de la conduite $= \int_{-80}^{0} \sqrt{1 + P'^2(x)}\,dx \overset{V200}{\simeq} 97{,}6530$ m

Prix de la conduite $C_1 \simeq 100 \cdot 97{,}6530 = 9\,765{,}30$ €

Aire du gazon $= \int_{-80}^{-\frac{2505}{71}} P(x)\,dx - \int_{-\frac{2505}{71}}^{0} P(x)\,dx \simeq 665{,}2642$ m²

prix du gazon $C_2 \simeq 12 \cdot 665{,}2642 \simeq 7983{,}17$ €

Coût total $= C_1 + C_2 \simeq 17\,748{,}47$ €

Rem : En utilisant les valeurs approchées $a \simeq 5 \cdot 10^{-4}$ et $b \simeq 1{,}8 \cdot 10^{-2}$ on obtient :
$c = -36$; $C_1 \simeq 9723{,}70$ €, $C_2 \simeq 7833{,}34$ €, coût total $\simeq 17\,557{,}04$ €

3) $M(50, 40)$, $N(x_N, 0)$, $MN = \sqrt{(50 - x_N)^2 + 40^2}$

4) coût de $[ON] = 30 \cdot x$ €
coût de $[NM] = 80 \cdot \sqrt{(50-x)^2 + 1600}$ €
coût total $C_2(x) = 30x + 80\sqrt{(50-x)^2 + 1600}$ €

5)

x	0		$33{,}82$		50
$C_2'(x)$		$-$	0	$+$	
$C_2(x)$		$\searrow$	$4466{,}48$	$\nearrow$	

($\to$ V200)

$C_2'(x) = 0 \overset{V200}{\iff} x \simeq 33{,}82$

$C_2(x)$ minimal pour $x \simeq 33{,}82$

coût minimal $\simeq 4466{,}48$ €

6) Chacun doit payer : $\dfrac{17748{,}47 + 4466{,}48}{3} \simeq 7404{,}98 \simeq \underline{7405}$ €