Mathématiques II – Epreuve écrite − Corrigé

I 1) a) $7^{x+\frac{4}{3}} - 5^{3x} = 2(7^{x+\frac{1}{3}} + 5^{3x-1})$

$\Leftrightarrow\ 7\cdot 7^{x+\frac{1}{3}} - 5^{3x} = 2\cdot 7^{x+\frac{1}{3}} + 2\cdot\frac{1}{5}\cdot 5^{3x}$

$\Leftrightarrow\ (7-2)\cdot 7^{x+\frac{1}{3}} = (\frac{2}{5}+1)\cdot 5^{3x}$

$\Leftrightarrow\ 5\cdot 7^{x+\frac{1}{3}} = \frac{7}{5}\cdot 5^{3x}$

$\Leftrightarrow\ 7^{x-\frac{2}{3}} = 5^{3x-2}$

$\Leftrightarrow\ e^{(x-\frac{2}{3})\ln 7} = e^{(3x-2)\ln 5}$

$\Leftrightarrow\ x\cdot(\ln 7 - 3\ln 5) = \frac{2}{3}\ln 7 - 2\ln 5$

$\Leftrightarrow\ x = \dfrac{\frac{2}{3}\ln 7 - 2\ln 5}{\ln 7 - 3\ln 5}$

$\Leftrightarrow\ x = \dfrac{\frac{2}{3}(\ln 7 - 3\ln 5)}{\ln 7 - 3\ln 5}$

$\Leftrightarrow\ x = \frac{2}{3}$

$S = \{\ \frac{2}{3}\ \}$

b) $\log_{x+2}(2x) = \log_{2x}(x+2)$

C.E. : 1) $x+2>0$ et $x+2\neq 1 \Leftrightarrow x>-2$ et $x\neq -1$

2) $2x>0$ et $2x\neq 1 \Leftrightarrow x>0$ et $x\neq\frac{1}{2}$

D = $]0\ ;\ \frac{1}{2}\ [\ \cup\ ]\frac{1}{2}\ ;\ +\infty[$

$\forall\, x\in D:\ \log_{x+2}(2x) = \log_{2x}(x+2) \Leftrightarrow \dfrac{\ln 2x}{\ln(x+2)} = \dfrac{\ln(x+2)}{\ln 2x}$

$\Leftrightarrow\ \ln^2 2x = \ln^2(x+2)$

$\Leftrightarrow\ \ln 2x = \ln(x+2)$ ou $\ln 2x = -\ln(x+2)$

$\Leftrightarrow\ 2x = x+2$ ou $2x = \frac{1}{x+2}$

$\Leftrightarrow\ x=2$ ou $2x^2+4x-1=0$ $[\Delta = 24]$

$\Leftrightarrow\ x=2$ ou $x=\frac{-2+\sqrt{6}}{2}$ ou $x=\frac{-2-\sqrt{6}}{2}$ $(\notin D)$

$S = \{\ \frac{-2+\sqrt{6}}{2}\ ;\ 2\}$

2) $(m+1)e^x - (m-1)e^{-x} = 2m \ \Big|\cdot e^x$ (E)

$\Leftrightarrow\ (m+1)e^{2x} - 2me^x - (m-1) = 0$ posons : $y = e^x > 0$

$\Leftrightarrow\ (m+1)y^2 - 2my - (m-1) = 0$

- $\boxed{m=-1}$
  (E) $\Leftrightarrow\ 2y+2=0 \Leftrightarrow y=-1$ impossible
  (E) n'admet aucune solution réelle.

- $\boxed{m\neq -1}$
  $\Delta = (-2m)^2 + 4(m+1)(m-1) = 4m^2+4m^2-4 = 8m^2-4$
  Produit des racines éventuelles : $P = \dfrac{c}{a} = -\dfrac{m-1}{m+1}$
  Somme des racines éventuelles : $S = \dfrac{-b}{a} = \dfrac{2m}{m+1}$

m	$-\infty$		-1		$-\frac{\sqrt{2}}{2}$		0		$\frac{\sqrt{2}}{2}$		1		$+\infty$
Δ		$+$	‖	$+$	0	$-$		$-$	0	$+$		$+$	
P		$-$	‖	$+$		$+$		$+$		$+$	0	$-$	
S		$+$	‖	$-$		$-$	0	$+$		$+$		$+$	
nombre de solutions en x de (E)		1	0	0	0	0	0	0	1	2	1	1	

Si $m \in [-1\,;\ \frac{\sqrt{2}}{2}[$, alors (E) n'admet aucune solution réelle.

Si $m \in]-\infty\,;\ -1[\ \cup \{\frac{\sqrt{2}}{2}\} \cup [1\,;\ +\infty[$, alors (E) admet exactement une solution réelle.

Si $m \in]\frac{\sqrt{2}}{2}\,;\ 1[$, alors (E) admet exactement deux solutions réelles.

II $f(x) = x \cdot \ln \dfrac{x+1}{x}$

1) *dom f* = $]-\infty\,;\ -1[\ \cup]0\,;\ +\infty[$ = *dom f '*

2) $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(x\cdot\ln\frac{x+1}{x}\right) = \lim_{x\to\pm\infty}\frac{\ln\frac{x+1}{x}\to 0}{\frac{1}{x}\to 0} \underset{H}{=} \lim_{x\to\pm\infty}\frac{\frac{x}{x+1}\cdot\frac{x-(x+1)}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\pm\infty}\frac{x}{x+1} = 1$$ A.H. : $y = 1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x\cdot\ln\frac{x+1}{x}\right) = \lim_{x\to 0^+}\frac{\ln\frac{x+1}{x}\to +\infty}{\frac{1}{x}\to +\infty} \underset{H}{=} \lim_{x\to 0^+}\frac{\frac{x}{x+1}\cdot\frac{x-(x+1)}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\frac{x}{x+1} = 0$$ « trou » au pt. O(0 ; 0)

$$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-}\left(\underbrace{x}_{\to -1}\cdot\underbrace{\ln\overbrace{\frac{x+1}{x}}^{\to 0^+}}_{\to -\infty}\right) = +\infty$$ A.V. : $x = -1$

3) a) $\forall\, x \in]-\infty\,;\ -1[\ \cup]0\,;\ +\infty[$: $f'(x) = 1\cdot\ln\dfrac{x+1}{x} + x\cdot\dfrac{x}{x+1}\cdot\dfrac{-1}{x^2} = \ln\dfrac{x+1}{x} - \dfrac{1}{x+1}$

b) $$\lim_{x\to\pm\infty} f'(x) = \lim_{x\to\pm\infty}\left(\underbrace{\ln\overbrace{\frac{x+1}{x}}^{\to 1}}_{\to 0} - \underbrace{\frac{1}{x+1}}_{\to 0}\right) = 0$$

$\forall\, x \in]-\infty\,;\ -1[\ \cup]0\,;\ +\infty[$: $f''(x) = \dfrac{x}{x+1}\cdot\dfrac{-1}{x^2} - \dfrac{-1}{(x+1)^2} = \dfrac{-x(x+1)+x^2}{x^2(x+1)^2} = \dfrac{-x}{x^2(x+1)^2}$

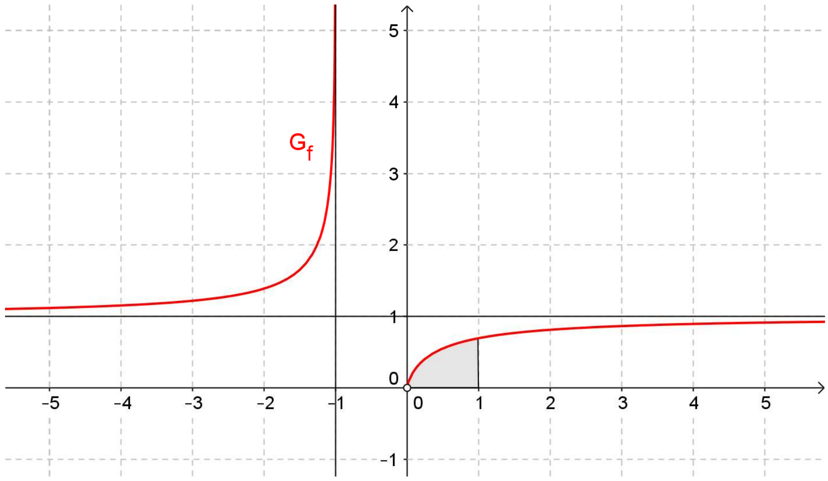
x	$-\infty$		-1		0		$+\infty$
$f''(x)$		$+$	‖	//////	‖	$-$	
f'	0	↗	‖	//////	‖	↘	0

c) On peut en déduire que $\forall\, x \in]-\infty\,;\ -1[\ \cup]0\,;\ +\infty[$: $f'(x) > 0$

d)

x	$-\infty$		-1		0		$+\infty$
$f'(x)$		$+$	‖	//////	‖	$+$	
$f''(x)$		$+$	‖	//////	‖	$-$	
f	1	↗	$+\infty$ ‖	//////	‖ 0	↗	1
G_f	A.H.G.	∪	A.V. ‖	//////	‖ "trou"	∩	A.H.D.

4)



5) Soit $M(m; f(m))$ $(m \in dom\, f)$ le point cherché.

$t_m \equiv y - f(m) = f'(m)\cdot(x-m)$

$$P(0\,;2)\in t_m \Leftrightarrow 2 - m\cdot\ln\frac{m+1}{m} = \left(\ln\frac{m+1}{m} - \frac{1}{m+1}\right)\cdot(0-m)$$

$$\Leftrightarrow 2 - m\cdot\ln\frac{m+1}{m} = -m\ln\frac{m+1}{m} + \frac{m}{m+1}$$

$$\Leftrightarrow 2 = \frac{m}{m+1}$$

$$\Leftrightarrow m = 2m+2$$

$$\Leftrightarrow m = -2 \qquad M(-2;2\ln 2)$$

6) $A(\alpha) = \int_\alpha^1 x\cdot\ln\frac{x+1}{x}\,dx$

$$\left\| \quad f(x) = \ln\frac{x+1}{x} \qquad f'(x) = \frac{x}{x+1}\cdot\frac{-1}{x^2} \qquad g'(x) = x \qquad g(x) = \frac{x^2}{2}\right.$$

$$\underset{IPP}{=}\left[\frac{x^2}{2}\ln\frac{x+1}{x}\right]_\alpha^1 + \frac{1}{2}\int_\alpha^1 \frac{x}{x+1}\,dx$$

$$= \left(\frac{1}{2}\ln 2 - \frac{\alpha^2}{2}\ln\frac{\alpha+1}{\alpha}\right) + \frac{1}{2}\int_\alpha^1\left(\frac{x+1}{x+1} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\left(\ln 2 - \alpha^2\ln(\alpha+1) + \alpha^2\ln\alpha + \left[x - \ln|x+1|\right]_\alpha^1\right)$$

$$= \frac{1}{2}\left(\ln 2 - \alpha^2\ln(\alpha+1) + \alpha^2\ln\alpha + 1 - \ln 2 - \alpha + \ln(\alpha+1)\right)$$

$$= \frac{1}{2}\left((1-\alpha^2)\ln(\alpha+1) + \alpha^2\ln\alpha + 1 - \alpha\right)$$

$$\lim_{\alpha\to 0^+} A(\alpha) = \lim_{\alpha\to 0^+}\frac{1}{2}\left(\underbrace{(1-\alpha^2)\ln(\alpha+1)}_{\to 0} + \underbrace{\alpha^2\ln\alpha}_{\to 0} + \underbrace{1-\alpha}_{\to 1}\right)$$

$$= \frac{1}{2} \text{ u.a.}$$

$$\left\| \quad \lim_{\alpha\to 0^+}\alpha^2\ln\alpha = \lim_{\alpha\to 0^+}\frac{\overset{\to -\infty}{\ln\alpha}}{\underset{\to +\infty}{\frac{1}{\alpha^2}}} \underset{H}{=} \lim_{\alpha\to 0^+}\frac{\frac{1}{\alpha}}{-\frac{2\alpha}{\alpha^4}} = \lim_{\alpha\to 0^+}\left(-\frac{\alpha^2}{2}\right) = 0\right.$$

III $f(x)=\begin{cases}\dfrac{x}{1+e^{\frac{1}{x}}} & \text{si } x\neq 0\\ 0 & \text{si } x=0\end{cases}$ $dom\, f=\mathbb{R}$

1) $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{x \ \to 0}{1+e^{\frac{1}{x}} \to +\infty}=0$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{x \ \to 0}{1+e^{\frac{1}{x}} \to 1}=0$

$\lim\limits_{x\to 0} f(x)=0=f(0)$; donc *f* est continu en 0.

$\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{1}{1+e^{\frac{1}{x}} \to +\infty}=0=f'_d(0)$

$\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{1}{1+e^{\frac{1}{x}} \to 1}=1=f'_g(0)$

$f'_g(0)=1\neq 0=f'_d(0)$; donc *f* n'est pas dérivable en 0 et O(0 ; 0) est un point anguleux du graphe de *f*.

2) $\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{x \ \to \pm\infty}{1+e^{\frac{1}{x}} \to 2}=\pm\infty$ pas d'A.H.

$\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{1 \ \to 1}{1+e^{\frac{1}{x}} \to 2}=\dfrac{1}{2}$

$\lim\limits_{x\to\pm\infty}\left[f(x)-\dfrac{1}{2}x\right]=\lim\limits_{x\to\pm\infty}\left(\dfrac{x}{1+e^{\frac{1}{x}}}-\dfrac{x}{2}\right)=\lim\limits_{x\to\pm\infty}\dfrac{x(1-e^{\frac{1}{x}}) \to -1 (*)}{2(1+e^{\frac{1}{x}}) \to 4}=-\dfrac{1}{4}$

$(*)\ \lim\limits_{x\to\pm\infty}\underbrace{x}_{\to\pm\infty}\underbrace{(1-e^{\frac{1}{x}})}_{\to 0}=\lim\limits_{x\to\pm\infty}\dfrac{1-e^{\frac{1}{x}} \to 0}{\frac{1}{x} \to 0}\underset{H}{=}\lim\limits_{x\to\pm\infty}\dfrac{\frac{1}{x^2}e^{\frac{1}{x}}}{-\frac{1}{x^2}}=\lim\limits_{x\to\pm\infty}(-e^{\frac{1}{x}})=-1$

G_f admet une A.O. d'équation $y=\dfrac{1}{2}x-\dfrac{1}{4}$.

IV 1) $\int_0^{\frac{\pi}{2}} \cos x \ln(1+\cos x)\,dx$

$f(x) = \ln(1+\cos x)$ $\qquad f'(x) = \frac{1}{1+\cos x}\cdot(-\sin x)$

$g'(x) = \cos x$ $\qquad g(x) = \sin x$

$$\underset{IPP}{=} \left[\sin x \ln(1+\cos x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x}\,dx$$

$$= (0-0) + \int_0^{\frac{\pi}{2}} \frac{1-\cos^2 x}{1+\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos x)\,dx$$

$$= \left[x - \sin x\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{\pi}{2} - 1$$

2) $f(x) = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$ $\qquad \forall\, x \in\,]1\,;+\infty[$

$\Leftrightarrow\ \frac{1}{x^4-1} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}$ $\qquad \forall\, x \in\,]1\,;+\infty[$

$\Leftrightarrow\ 1 = a(x+1)(x^2+1) + b(x-1)(x^2+1) + (cx+d)(x^2-1)$ $\qquad \forall\, x \in\,]1\,;+\infty[$

$\Leftrightarrow\ 1 = a(x^3+x^2+x+1) + b(x^3-x^2+x-1) + (cx^3+dx^2-cx-d)$ $\qquad \forall\, x \in\,]1\,;+\infty[$

$$\Leftrightarrow \begin{cases} a+b+c=0 \ (I) \\ a-b+d=0 \ (II) \\ a+b-c=0 \ (III) \\ a-b-d=1 \ (IV) \end{cases} \underset{\substack{(I)-(III)\\(II)-(IV)\\(I)\\(II)}}{\Leftrightarrow} \begin{cases} c=0 \\ d=-\frac{1}{2} \\ a+b=0 \ (V) \\ a-b=\frac{1}{2} \ (VI) \end{cases} \underset{(V)+(VI)}{\Leftrightarrow} \begin{cases} c=0 \\ d=-\frac{1}{2} \\ b=-a \\ 2a=\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} a=\frac{1}{4} \\ b=-\frac{1}{4} \\ c=0 \\ d=-\frac{1}{2} \end{cases}$$

$\forall\, x \in\,]1\,;+\infty[\ :\ f(x) = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$

Sur $]1\,;+\infty[$: $\int f(x)\,dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\mathit{Arc}\tan x + c \quad (c \in \mathbb{R})$

$$= \underbrace{\frac{1}{4}\ln\frac{x-1}{x+1} - \frac{1}{2}\mathit{Arc}\tan x + c}_{F(x)}$$

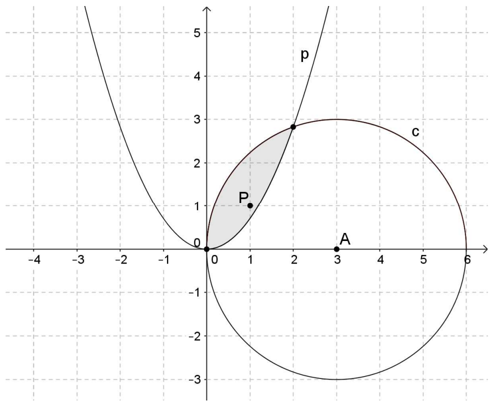
$F(\sqrt{3}) = 0 \ \Leftrightarrow\ \frac{1}{4}\ln\frac{\sqrt{3}-1}{\sqrt{3}+1} - \frac{1}{2}\mathit{Arc}\tan\sqrt{3} + c = 0$

$\Leftrightarrow\ \frac{1}{4}\ln(2-\sqrt{3}) - \frac{1}{2}\cdot\frac{\pi}{3} + c = 0$

$\Leftrightarrow\ c = -\frac{1}{4}\ln(2-\sqrt{3}) + \frac{\pi}{6}$

$F(x) = \frac{1}{4}\ln\frac{x-1}{x+1} - \frac{1}{2}\mathit{Arc}\tan x - \frac{1}{4}\ln(2-\sqrt{3}) + \frac{\pi}{6}$

V **1)** $c \cap p \equiv \begin{cases} (x-3)^2 + y^2 = 3^2 & (I) \\ y = \frac{1}{\sqrt{2}} x^2 & (II) \end{cases}$



(II) dans (I) : $x^2 - 6x + 9 + \frac{1}{2} x^4 = 9$

$\Leftrightarrow \; x^4 + 2x^2 - 12x = 0$

$\Leftrightarrow \; x \cdot \underbrace{(x^3 + 2x - 12)}_{P(x)} = 0 \qquad \| \; P(2) = 0$

$\Leftrightarrow \; x \cdot (x-2)(x^2 + 2x + 6) = 0 \qquad \| \; \Delta = -20 < 0$

$\Leftrightarrow \; x = 0$ ou $x = 2$

2) Aire demandée : $\displaystyle\int_0^2 \left(\sqrt{9 - (x-3)^2} - \frac{1}{\sqrt{2}} x^2 \right) dx$

Calculons:

$$\int \sqrt{9-(x-3)^2}\, dx = 3\int \sqrt{1 - \left(\frac{x-3}{3}\right)^2}\, dx$$

$$= 9 \int \sqrt{1 - \sin^2 t} \, \cos t \, dt$$

$$= 9 \int \sqrt{\cos^2 t}\, \cos t \, dt$$

$$= 9 \int \cos^2 t dt$$

$$= 9 \int \left(\tfrac{1}{2} + \tfrac{1}{2} \cos 2t \right) dt$$

$$= \tfrac{9}{2} \left(t + \tfrac{1}{2} \sin 2t \right) + c \quad (c \in \mathbb{R})$$

$$= \tfrac{9}{2} (t + \sin t \cos t) + c$$

$$= \frac{9}{2} \left(Arc \sin \frac{x-3}{3} + \frac{x-3}{3} \sqrt{1 - \left(\frac{x-3}{3} \right)^2} \right) + c$$

$$= \frac{9}{2} Arc \sin \frac{x-3}{3} + \frac{x-3}{2} \sqrt{9 - (x-3)^2} + c$$

posons : $t = Arc\sin\left(\frac{x-3}{3}\right) \Leftrightarrow \frac{x-3}{3} = \sin t$

$dx = 3 \cos t \, dt$

$\sqrt{\cos^2 t} = |\cos t| = \cos t$ car $\cos t \geq 0$ $(t \in [-\frac{\pi}{2}; \frac{\pi}{2}])$

Donc : $\displaystyle\int_0^2 \left(\sqrt{9-(x-3)^2} - \frac{1}{\sqrt{2}} x^2 \right) dx$

$$= \left[\frac{9}{2} Arc \sin \frac{x-3}{3} + \frac{x-3}{2} \sqrt{9-(x-3)^2} - \frac{\sqrt{2}}{6} x^3 \right]_0^2$$

$$= \frac{9}{2} Arc \sin \frac{-1}{3} - \frac{1}{2} \cdot 2\sqrt{2} - \frac{8\sqrt{2}}{6} + \frac{9\pi}{4}$$

$$= \frac{9\pi}{4} - \frac{7\sqrt{2}}{3} - \frac{9}{2} Arc \sin \frac{1}{3}$$

$\approx 2{,}24$ u.a.

VI Volume demandé : $V=\pi\int_0^{\pi}[f(x)]^2dx=\pi\int_0^{\pi}\sin^2(x)\cdot e^x\,dx$

$F(x)=\int\sin^2(x)\cdot e^x\,dx$ $\qquad\left\|\begin{array}{ll} u(x)=\sin^2 x & u'(x)=2\sin x\cos x=\sin 2x \\ v'(x)=e^x & v(x)=e^x \end{array}\right.$

$\underset{IPP}{=}\sin^2(x)\cdot e^x-\int\sin(2x)\cdot e^x dx$ $\qquad\left\|\begin{array}{ll} u(x)=\sin 2x & u'(x)=2\cos 2x \\ v'(x)=e^x & v(x)=e^x \end{array}\right.$

$\underset{IPP}{=}\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2\int\cos(2x)\cdot e^x\,dx$

$=\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2\int(1-2\sin^2 x)\cdot e^x\,dx$

$=\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2e^x-4F(x)$

$5F(x)=\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2e^x+k\quad(k\in\mathbb{R})$

$F(x)=\frac{1}{5}\left(\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2e^x\right)+c\quad(c\in\mathbb{R})$

Donc : $V=\pi\cdot\left[\frac{1}{5}\left(\sin^2(x)\cdot e^x-\sin(2x)\cdot e^x+2e^x\right)\right]_0^{\pi}=\pi\cdot\left(\frac{2}{5}e^{\pi}-\frac{2}{5}\right)=\frac{2\pi(e^{\pi}-1)}{5}\approx 27{,}823\,u.v.$
